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Sequences of Levy transformations and multi-Wroński determinant solutions of the Darboux system

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Abstract

Sequences of Levy transformations for the Darboux system of conjugates nets in multidimensions are studied. We show that after a suitable number of Levy transformations, with at least a Levy transformation in each direction, we get closed formulae in terms of multi-Wroński determinants. These formulae are for the tangent vectors, Lamé coefficients, rotation coefficients and points of the surface. © 1998 Elsevier Science B.V.

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1.

The interaction between soliton theory and geometry is a growing subject. In fact, many systems that appear by geometrical considerations have been studied independently in soliton theory; well-known examples include the Liouville and sine-Gordon equations which characterize minimal and pseudo-spherical surfaces, respectively. Another relevant case is given by the Darboux equations that were solved 12 years ago in its matrix generalization, using the $\bar{\partial}$-dressing, by Zakharov and Manakov [14].

In this note we want to iterate a transformation that preserves the Darboux equations which is known as Levy transformation [10].

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2.

The Darboux equations

$$\frac{\partial \beta_{ij}}{\partial u_k} = \beta_{ik}\beta_{kj}, \quad i, j, k = 1, \dots, N, \quad i \neq j \neq k \neq i, \tag{1}$$

for the $N(N-1)$ functions $\{\beta_{ij}\}_{i,j=1,\dots,N;i\neq j}$ of $u_1, \dots, u_N$, characterize N-dimensional submanifolds of $\mathbb{R}^P$, $N \leq P$, parametrized by conjugate coordinate systems [2,4], and are the compatibility conditions of the following linear system:

$$\frac{\partial \boldsymbol{X}_j}{\partial u_i} = \beta_{ji}\boldsymbol{X}_i, \quad i, j = 1, \dots, N, \quad i \neq j, \tag{2}$$

involving suitable P-dimensional vectors $\boldsymbol{X}_i$, tangent to the coordinate lines. The so-called Lamé coefficients satisfy

$$\frac{\partial H_j}{\partial u_i} = \beta_{ij} H_i, \quad i, j = 1, \dots, N, \quad i \neq j, \tag{3}$$

and the points of the surface $\boldsymbol{x}$ can be found by means of

$$\frac{\partial \boldsymbol{x}}{\partial u_i} = \boldsymbol{X}_i H_i, \quad i = 1, \dots, N, \tag{4}$$

which is equivalent to the more standard Laplace equation

$$\frac{\partial^2 \boldsymbol{x}}{\partial u_i \partial u_j} = \frac{\partial \ln H_i}{\partial u_j}\frac{\partial \boldsymbol{x}}{\partial u_i} + \frac{\partial \ln H_j}{\partial u_i}\frac{\partial \boldsymbol{x}}{\partial u_j}, \quad i, j = 1, \dots, N, \quad i \neq j.$$

A Darboux type transformation for this system was found by Levy [5,9,10]. In fact, in [10] the transformation is constructed only for two-dimensional surfaces, $N = 2$, being the Darboux equations in this case trivial and Levy only presents the transformation for the points of the surface. However, in [9] the Levy transformation is extended to the first nontrivial case of Darboux equations, namely $N = 3$. The extension to arbitrary N is straightforward and reads as follows. Given a solution ξ_j of

$$\frac{\partial \xi_j}{\partial u_k} = \beta_{jk}\xi_k$$

for each of the N possible directions in the coordinate space there is a corresponding Levy transformation that reads for the ith case:

$$\boldsymbol{x}[1] = \boldsymbol{x} - \frac{\Omega(\xi, H)}{\xi_i}\boldsymbol{X}_i,$$

$$\boldsymbol{X}_i[1] = \frac{1}{\xi_i}\left(\xi_i \frac{\partial \boldsymbol{X}_i}{\partial u_i} - \frac{\partial \xi_i}{\partial u_i}\boldsymbol{X}_i\right), \qquad \boldsymbol{X}_k[1] = \frac{1}{\xi_i}(\xi_i \boldsymbol{X}_k - \xi_k \boldsymbol{X}_i),$$

$$H_i[1] = -\frac{\Omega(\xi, H)}{\xi_i}, \qquad H_k[1] = H_k - \beta_{ik}\frac{\Omega(\xi, H)}{\xi_i},$$

$$\beta_{ik}[1] = -\frac{1}{\xi_i}\left(\beta_{ik}\frac{\partial \xi_i}{\partial u_i} - \xi_i\frac{\partial \beta_{ik}}{\partial u_i}\right),$$

$$\beta_{ki}[1] = -\frac{\xi_k}{\xi_i}, \qquad \beta_{kl}[1] = -\frac{\xi_k\beta_{il} - \xi_i\beta_{kl}}{\xi_i},$$

where $k, l = 1, \ldots, N$ with $k \neq l \neq i$. Here we have introduced the potential $\Omega(\xi, H)$ defined by

$$\frac{\partial \Omega(\xi, H)}{\partial u_k} = \xi_k H_k, \quad k = 1, \ldots, N,$$

which are compatible equations by means of the equations satisfied by ξ_k and H_k.

3.

Using Crum type ideas [1] one can iterate this Levy transformation. However now there is a difference with respect to the iteration of the Darboux transformation of the one-dimensional Schrödinger equation: we have N different elementary Levy's transformations $\{\mathcal{L}_i\}_{i=1,\ldots,N}$.

If one performs less than N iterations or more than N iterations, say $\mathcal{L}_{i_1} \cdots \mathcal{L}_{i_M}$ with $\{1, \ldots, N\} \not\subset \{i_1, \ldots, i_M\}$, one gets nonsymmetric formulae in which the initial β's and its derivatives appear explicitly. However, if in the latter case we have $\{1, \ldots, N\} \subset \{i_1, \ldots, i_M\}$, that is we have performed at least one Levy transformation in each spatial direction we obtain formulae only in terms of Wroński determinants of the wave functions with no β's appearing explicitly.

To present our main result, we introduce some convenient notations. First we define $\partial_i := \partial/\partial u_i$. Second, for any set of functions $\{\xi_j^i\}_{i=1,\ldots,M;\, j=1,\ldots,N}$ we denote by $W_j(n)$ the following Wroński matrix:

$$W_j(n) := W_j(\xi_j^1, \ldots, \xi_j^M) := \begin{pmatrix} \xi_j^1 & \xi_j^2 & \cdots & \xi_j^M \\ \partial_j\xi_j^1 & \partial_j\xi_j^2 & \cdots & \partial_j\xi_j^M \\ \vdots & \vdots & & \vdots \\ \partial_j^{n-1}\xi_j^1 & \partial_j^{n-1}\xi_j^2 & \cdots & \partial_j^{n-1}\xi_j^M \end{pmatrix}.$$

For any partition of $M = m_1 + m_2 + \ldots + m_N$, we construct a multi-Wroński matrix

$$\mathcal{W} := \begin{pmatrix} W_1(m_1) \\ W_2(m_2) \\ \vdots \\ W_N(m_N) \end{pmatrix}.$$

Now we are ready to present the following:

Theorem. *Given M functions* $\{\xi_i^j\}_{i=1,\dots,N;\, j=1,\dots,M}$ *and* $X_i = (X_i^1, \dots, X_i^P)^t$, $i = 1, \dots, N$, *all of them solutions of* (2) *and* H_i, $i = 1, \dots, N$, *solutions of* (3), *for given* β_{ij}, *then new solutions* $X_i[M]$, $H_i[M]$ *and* $\beta_{ij}[M]$ *are defined by:*

$$X_i^l[M] = \frac{|\mathbb{X}_i^l|}{|\mathcal{W}|}, \qquad H_i[M] = -\frac{|\mathbb{H}_i|}{|\mathcal{W}|}, \qquad \beta_{ij}[M] = -\frac{|\mathcal{W}_{ij}|}{|\mathcal{W}|},$$

where

$$\mathbb{X}_i^l = \begin{pmatrix} \mathcal{W} & \boldsymbol{v}^l \\ \partial_i^{m_i}\boldsymbol{\xi}_i & \partial_i^{m_i} X_i^l \end{pmatrix},$$

with

$$\boldsymbol{v}^l := (\boldsymbol{v}_1^l, \dots, \boldsymbol{v}_N^l)^t, \quad \text{being } \boldsymbol{v}_k^l := (X_k^l, \partial_k X_k^l, \dots, \partial_k^{m_k-1} X_k^l),$$
$$\boldsymbol{\xi}_i := (\xi_i^1, \dots, \xi_i^M),$$

$\mathbb{H}_i$ *is obtained from* $\mathcal{W}$ *by replacing the last row of the* i*th block by* $\Omega(\boldsymbol{\xi}, H)$ *and* $\mathcal{W}_{ij}$ *by replacing the last row of the* j*th block by* $\partial_i^{m_i}\boldsymbol{\xi}_i$. *In the partition* $M = m_1 + m_2 + \dots + m_N$ *we need* $m_i \in \mathbb{N}$.

Moreover, for the transformed surface we have the parametrization

$$\boldsymbol{x}[M] = \frac{1}{|\mathcal{W}|}\left(\begin{vmatrix} \mathcal{W} & \boldsymbol{v}^1 \\ \Omega(\boldsymbol{\xi}, H) & x^1 \end{vmatrix}, \dots, \begin{vmatrix} \mathcal{W} & \boldsymbol{v}^P \\ \Omega(\boldsymbol{\xi}, H) & x^P \end{vmatrix}\right)^t.$$

Proof. The proof that follows is inspired by Refs. [6,11], however is extended to this multicomponent system and we give a more detailed account.

We first need to show that

$$\partial_k X_i^l[M] = \beta_{ik}[M] X_k^l[M],$$

or equivalently that the following bilinear equation holds:

$$|\mathcal{W}|\partial_k|\mathbb{X}_i^l| - |\mathbb{X}_i^l|\partial_k|\mathcal{W}| + |\mathbb{X}_k^l||\mathcal{W}_{ik}| = 0.$$

To this aim we consider the following $(2M+1) \times (2M+1)$ square matrix:

$$\mathcal{A}_{ik}^l := \begin{pmatrix} A_k & 0 & \partial_k^{m_k-1}\boldsymbol{\xi}_k^t & \partial_k^{m_k}\boldsymbol{\xi}_k^t & \partial_i^{m_i}\boldsymbol{\xi}_i^t \\ 0 & A_k & \partial_k^{m_k-1}\boldsymbol{\xi}_k^t & \partial_k^{m_k}\boldsymbol{\xi}_k^t & \partial_i^{m_i}\boldsymbol{\xi}_i^t \\ 0 & \boldsymbol{b}_k^l & \partial_k^{m_k-1} X_k^l & \partial_k^{m_k} X_k^l & \partial_i^{m_i} X_i^l \end{pmatrix},$$

where A_k is a $M \times (M-1)$ rectangular matrix

$$(A_k)^t := \begin{pmatrix} W_1(m_1) \\ \vdots \\ \hat{W}_k(m_k) \\ \vdots \\ W_N(m_N) \end{pmatrix},$$

with $\hat{W}_k(m_k)$ obtained from $W_k(m_k)$ by deleting the last row, and

$$\boldsymbol{b}_k^l = (\boldsymbol{v}_1^l, \dots, \hat{\boldsymbol{v}}_k^l, \dots, \boldsymbol{v}_N^l),$$

with $\hat{\boldsymbol{v}}_k^l$ obtained by deleting the last element in $\boldsymbol{v}_k^l$.

We now recall Laplace's general expansion theorem [12] that we shall use in this proof, this theorem allows the computation of an $n \times n$ matrix $A := (a_{ij})$ as follows:

$$\det A = \sum_{\substack{\rho_1,\dots,\rho_r \\ \rho_1<\dots<\rho_r}} (-1)^{\gamma_1+\dots+\gamma_r+\rho_1+\dots+\rho_r} \times \begin{vmatrix} a_{\gamma_1\rho_1} & a_{\gamma_1\rho_2} & \cdots & a_{\gamma_1\rho_r} \\ a_{\gamma_2\rho_1} & a_{\gamma_2\rho_2} & \cdots & a_{\gamma_2\rho_r} \\ \vdots & \vdots & & \vdots \\ a_{\gamma_r\rho_1} & a_{\gamma_r\rho_2} & \cdots & a_{\gamma_r\rho_r} \end{vmatrix} \times \begin{vmatrix} a_{\delta_1\sigma_1} & a_{\delta_1\sigma_2} & \cdots & a_{\delta_1\sigma_s} \\ a_{\delta_2\sigma_1} & a_{\delta_2\sigma_2} & \cdots & a_{\delta_2\sigma_s} \\ \vdots & \vdots & & \vdots \\ a_{\delta_s\sigma_1} & a_{\delta_s\sigma_2} & \cdots & a_{\delta_s\sigma_s} \end{vmatrix},$$

where $r + s = n$ and

$$(\gamma_1, \dots, \gamma_r, \delta_1, \dots, \delta_s) = (1, \dots, n), \qquad (\rho_1, \dots, \rho_r, \sigma_1, \dots, \sigma_s) = (1, \dots, n),$$

up to permutations.

Let us now expand the determinant of the matrix $\mathcal{A}_{ik}^l$ by means of Laplace's general expansion theorem. Here, we take $r = M$, $\gamma_i = i$ and $\delta_i = M + i (i = 1, \dots, M)$. It is easy to see that

$$\begin{aligned} \left|\mathcal{A}_{ik}^l\right| = (-1)^{M-1} \Bigg(& |A_k \quad \partial_k^{m_k-1}\boldsymbol{\xi}_k^t| \times \begin{vmatrix} A_k & \partial_k^{m_k}\boldsymbol{\xi}_k^t & \partial_i^{m_i}\boldsymbol{\xi}_i^t \\ \boldsymbol{b}_k^l & \partial_k^{m_k}X_k^l & \partial_i^{m_i}\xi_i \end{vmatrix} \\ & - |A_k \quad \partial_k^{m_k}\boldsymbol{\xi}_k^t| \times \begin{vmatrix} A_k & \partial_k^{m_k-1}\boldsymbol{\xi}_k^t & \partial_i^{m_i}\boldsymbol{\xi}_i^t \\ \boldsymbol{b}_k^l & \partial_k^{m_k-1}X_k^l & \partial_i^{m_i}\xi_i \end{vmatrix} \\ & + |A_k \quad \partial_i^{m_i}\boldsymbol{\xi}_i^t| \times \begin{vmatrix} A_k & \partial_k^{m_k-1}\boldsymbol{\xi}_k^t & \partial_i^{m_k}\boldsymbol{\xi}_i^t \\ \boldsymbol{b}_k^l & \partial_k^{m_k-1}X_k^l & \partial_k^{m_k}\xi_k \end{vmatrix} \Bigg), \end{aligned}$$

expression that after an even number of permutations of columns and transposition reads

$$|\mathcal{A}_{ik}^l| = (-1)^{M-1}[|\mathcal{W}|\partial_k|\mathbb{X}_i^l| - |\mathbb{X}_i^l|\partial_k|\mathcal{W}| + |\mathbb{X}_k^l||\mathcal{W}_{ik}|].$$

But Laplace's theorem also implies $|\mathcal{A}_{ik}^l| = 0$, to see this we just use the standard version of this theorem and expand the determinant with respect to its last row. In doing so we get a sum in which all terms vanish, this last statement follows again from Laplace's general expansion theorem. This gives the desired result.

Next we prove that

$$\partial_k H_i[M] = \beta_{ki}[M] H_k[M],$$

or equivalently that the following bilinear equation holds

$$|\mathcal{W}|\partial_k|\mathbb{H}_i| - |\mathbb{H}_i|\partial_k\mathcal{W}| + |\mathcal{W}_{ki}||\mathbb{H}_k| = 0.$$

As before this relation is a consequence of Laplace's general expansion theorem. For this aim we consider the $2M \times 2M$ square matrix:

$$\mathcal{B}_{ik} := \begin{pmatrix} B_{ik} & 0 & \partial_k^{m_k-1}\xi_k^t & \partial_k^{m_k}\xi_k^t & \partial_i^{m_i-1}\xi_i^t & \Omega(\xi, H)^t \\ 0 & B_{ik} & \partial_k^{m_k-1}\xi_k^t & \partial_k^{m_k}\xi_k^t & \partial_i^{m_i-1}\xi_i^t & \Omega(\xi, H)^t \end{pmatrix},$$

where B_{ik} is a $M \times (M-2)$ rectangular matrix

$$(B_{ik})^t := \begin{pmatrix} W_1(m_1) \\ \vdots \\ \hat{W}_i(m_i) \\ \vdots \\ \hat{W}_k(m_k) \\ \vdots \\ W_N(m_N) \end{pmatrix}.$$

Using the version of the Laplace expansion appearing in [6, Eq. (3.3)] we get the desired bilinear formula.

Finally, we prove the formula for $x^l[M] = \Omega(X^l[M], H[M])$ (see 4). This is achieved by considering the following $(2M+1) \times (2M+1)$ square matrix:

$$\mathcal{C}_{kk}^l := \begin{pmatrix} A_k & 0 & \partial_k^{m_k-1}\xi_k^t & \partial_k^{m_k}\xi_k^t & \Omega(\xi, H)^t \\ 0 & A_k & \partial_k^{m_k-1}\xi_k^t & \partial_k^{m_k}\xi_k^t & \Omega(\xi, H)^t \\ 0 & b_k^l & \partial_k^{m_k-1}X_k^l & \partial_k^{m_k}X_k^l & \Omega(X^l, H) \end{pmatrix},$$

and using that $x^l = \Omega(X^l, H)$ and Laplace's general expansion theorem. □

4.

Sequences of Levy transformations for two-dimensional surfaces have already been studied in [7,13] see also [5]. Let us remark that the Darboux equations are trivial in this case and that they only consider the points in the surface. Up to a factor $((H_1 \ldots H_N|\mathcal{W}|)^{-1})$ and the choice $H_i\xi_i^j = \partial_i\theta^{(j)}$ our formula for the points of the surface coincides, when $N = 2$, with the formula of [7], which, to our knowldege, is the first place where double wronskian appeared.

From a complete different point of view Nimmo considered in [11] what he called Darboux transformations for the two-dimensional Zakharov–Shabat/AKNS spectral problem, i.e. in the context of the Davey–Stewartson equations. In fact, this is intimately connected with two-dimensional conjugate nets [8]. His results are special cases of ours: first we have arbitrary dimension, not only $N = 2$ as in [11]; second our partition for $N = 2$, $M = m_1 + m_2$ is more general than his, $M = 2m$; third we have computed not only the transformation for the potentials $\beta_{12} = q$ and $\beta_{21} = r$ and wave functions but also for the adjoint wave function and for the corresponding points in the surface, that in this case, as we mentioned in the previous paragraph, can be found in [7].

The above remarks illustrate the fact that same problem has been tackled by different techniques coming from geometry on one the hand and soliton theory on the other, covering different aspects of it. In this paper we have extended the results of both approaches to higher dimensions, where the Darboux equations are no more trivial. In fact, from the soliton theory's point of view the Levy transformation for the Darboux system can be considered as an *elementary Darboux transformations* for the N-component Kadomtsev–Petviashvili hierarchy [3].

We have already mentioned that there exist other possibilities for iterations. In fact we have requested that there is at least one Levy transformation per direction. If this is not the case our results do not hold any more. However, one could get closed formulae where the β's appear explicitly. In principle, one has N possible *different* types of formulae. But we are not going to consider this problem in this letter.

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